Generalized Hermite Reduction, Creative Telescoping, and Definite Integration of D-Finite Functions

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Rencontres Arithmétiques du GDR Informatique Mathématique

Based on joint work with A. Bostan, P. Lairez, and B. Salvy

1. Motivation

2. Creative telescoping and Chyzak's algorithm for D-finite functions (Chyzak, 2000)

3. Hermite reduction and the definite integration of rational functions (Bostan, Chen, Chyzak, Li, 2010)

4. Generalized Hermite reduction and the definite integration of D-finite functions (Bostan, Chyzak, Lairez, Salvy, 2018)

5. Conclusion

Differentiating under the Integral Sign

Zeilberger's derivation (1982) of a classical integral

Given $f(b,x) = e^{-x^2} \cos 2bx$, find $F(b) = \int_{-\infty}^{+\infty} f(b,x)\,dx = ?$.

$$\frac{dF}{db}(b) = \int_{-\infty}^{+\infty} -2xe^{-x^2} \sin 2bx\,dx =$$

$$\left[e^{-x^2} \sin 2bx\right]_{x=-\infty}^{x=+\infty} + \int_{-\infty}^{+\infty} -2be^{-x^2} \cos 2bx\,dx = -2b\,F(b).$$

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Continuous form of **"Creative Telescoping"**:

$$\frac{df}{db}(b,x) + 2b\,f(b,x) = \frac{dg}{dx}(b,x) \quad \text{for} \quad g(b,x) = -\frac{1}{2x}\frac{df}{db}(b,x).$$

After integration over x from $-\infty$ to $+\infty$:

$$\frac{dF}{db}(b) + 2bF(b) = \left[\frac{dg}{dx}(b,x)\right]_{x=-\infty}^{x=+\infty} = 0 - 0 = 0.$$

Hermite Reduction (1872)

$$EA - \alpha FA' = P \implies \int \frac{P}{A^{\alpha+1}} = \frac{F}{A^\alpha} + \int \frac{E + F'}{A^\alpha}$$

Cela posé, l'intégrale $\int \frac{P\,dx}{A^{\alpha+1}}$ se traitera comme il suit : nous effectuerons sur A et sa dérivée A′ les opérations du plus grand commun diviseur, de manière à obtenir deux polynômes G et H, satisfaisant à la condition

$$AG - A'H = 1.$$

Nous formerons ensuite deux séries de fonctions entières :

$$V_0,\ V_1, \dots,\ V_{\alpha-1},$$
$$P_1,\ P_2, \dots,\ P_\alpha,$$

par ces relations, où les polynômes Q, Q_1, Q_2,... sont entièrement arbitraires, savoir :

$$\begin{aligned} \alpha V_0 &= HP - AQ, \\ (\alpha-1)V_1 &= HP_1 - AQ_1, \\ (\alpha-2)V_2 &= HP_2 - AQ_2, \\ &\dots\dots\dots\dots, \\ V_{\alpha-1} &= HP_{\alpha-1} - AQ_{\alpha-1}, \end{aligned}$$

$$\begin{aligned} P_1 &= GP - A'Q - V'_0, \\ P_2 &= GP_1 - A'Q_1 - V'_1, \\ &\dots\dots\dots\dots, \\ P_\alpha &= GP_{\alpha-1} - A'Q_{\alpha-1} - V'_{\alpha-1}. \end{aligned}$$

Maintenant je prouverai qu'en faisant

$$V = V_0 + AV_1 + A^2V_2 + \dots + A^{\alpha-1}V_{\alpha-1},$$
$$U = P_\alpha,$$

on a l'égalité

$$\frac{P}{A^{\alpha+1}} = \frac{U}{A} + \left(\frac{V}{A^\alpha}\right)';$$

d'où

$$\int \frac{P\,dx}{A^{\alpha+1}} = \int \frac{U\,dx}{A} + \frac{V}{A^\alpha},$$

de sorte que $\frac{V}{A^\alpha}$ est la partie algébrique de l'intégrale proposée, et $\int \frac{U\,dx}{A}$ la partie transcendante.

A cet effet, j'élimine G et H entre les trois égalités

$$\begin{aligned} AG - A'H &= 1, \\ (\alpha - i)V_i &= HP_i - AQ_i, \\ P_{i+1} &= GP_i - A'Q_i - V'_i, \end{aligned}$$

ce qui donne

$$AP_{i+1} = P_i + (\alpha - i)A'V_i - AV'_i.$$

Or on peut écrire cette relation de la manière suivante :

$$\frac{P_i}{A^{\alpha-i+1}} - \frac{P_{i+1}}{A^{\alpha-i}} = \left(\frac{V_i}{A^{\alpha-i}}\right)'.$$

En supposant ensuite $i = 0, 1, 2, \dots, \alpha - 1$ et ajoutant membre à membre, nous en conclurons

$$\frac{P}{A^{\alpha+1}} - \frac{P_\alpha}{A} = \left(\frac{V_0}{A^\alpha} + \frac{V_1}{A^{\alpha-1}} + \dots + \frac{V_{\alpha-1}}{A}\right)',$$

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comme il s'agissait de le démontrer.

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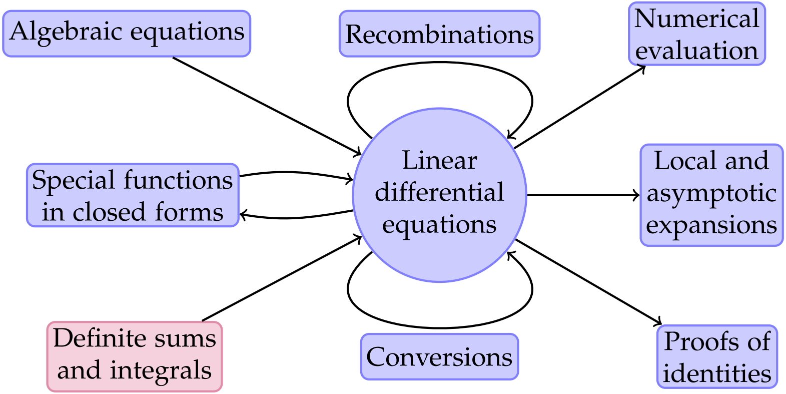
See also (Ostrogradsky, 1838, 1845).

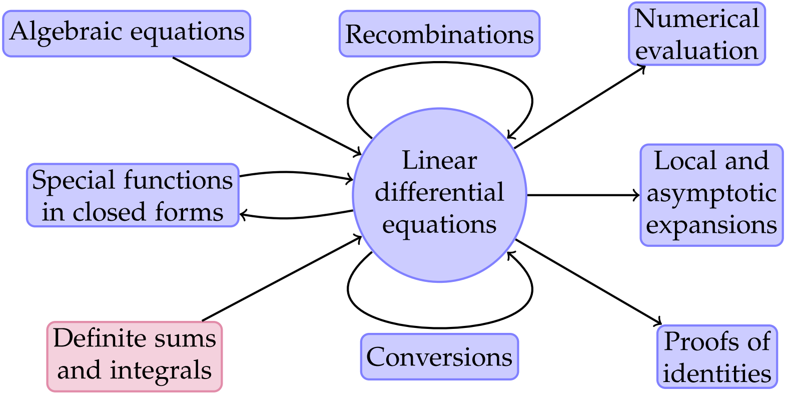
Linear Differential Equations as a Data Structure



Def: differentially finite functions (a.k.a. D-finite)

A function $f(x)$ is D-finite if its derivatives $f(x), f'(x), f''(x), \dots,$ span a finite-dimensional vector space over $\mathbb{C}(x)$.

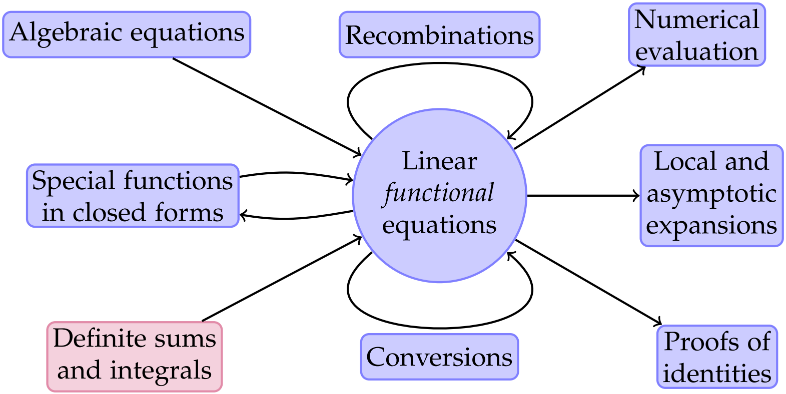
Linear Differential Equations as a Data Structure



Def: multivariate D-finite functions

A function $f(x,y,z)$ is D-finite iff its derivatives $\frac{\partial^{i+j+k} f}{\partial_x^i \partial_y^j \partial_z^k}(x,y,z)$, $i,j,k \geq 0$, span a finite-dimensional vector space over $\mathbb{C}(x,y,z)$.

Linear Differential Equations as a Data Structure

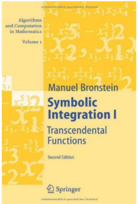


Def: multivariate ∂-finite functions

A function $f_{n,m}(x,y,z)$ is ∂-finite iff a similar confinement holds for derivatives, shifts, etc.

Symbolic Integration: Indefinite vs Definite

Risch's algorithm (1968)	Zeilberger's algorithm (1991)



(Bronstein, 1997)

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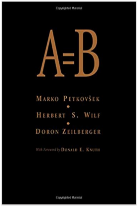


(Petkovšek, Wilf, Zeilberger, 1996)

$$\int \ln(1 - \exp(1 - \sqrt{\ln x - x})) = ?$$

$$\int_0^\infty x \exp(-x^2/p) J_n(x)\, dx = ?$$

Sums and Integrals (1/7)

Binomial sums and variations

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2 \qquad \text{(Blodgelt, 1990)}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \qquad \text{(Strehl, 1994)}$$

No explicit form, but a 2nd-order linear recurrence (Apéry, 1979):

$$\sum_{m=1}^{n}\frac{1}{m^3} + \sum_{m=1}^{k}\frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}$$

Sums and Integrals (2/7)

Integrals of the theory of special functions

Four types of Bessel functions (Glasser & Montaldi, 1994):

$$\int_0^{+\infty} x\, J_1(ax)\, I_1(ax)\, Y_0(x)\, K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$

No explicit form, but a 2nd-order linear ODE:

$$\int_0^{\infty} \int_0^{\infty} J_1(x)\, J_1(y)\, J_2(c\sqrt{xy})\, \frac{dx\, dy}{e^{x+y}}$$

Sums and Integrals (3/7)

Extractions of coefficients

Theory of orthogonal polynomials, here, Hermite (Doetsch, 1930):

$$\frac{1}{2\pi i} \oint \frac{(1 + 2xy + 4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1 + 4y^2)^{\frac{3}{2}}} \, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor!}$$

Scalar products involving orthogonal/parametrised families

Chebyshev polynomials, Bessel functions, modified Bessel functions:

$$\int_{-1}^{+1} \frac{e^{-px} T_n(x)}{\sqrt{1 - x^2}} \, dx = (-1)^n \pi \, I_n(p)$$

$$\int_0^{+\infty} x e^{-px^2} J_n(bx) I_n(cx) \, dx = \frac{1}{2p} \exp\left(\frac{c^2 - b^2}{4p}\right) J_n\left(\frac{bc}{2p}\right)$$

Sums and Integrals (4/7)

Diagonals in combinatorics

Generating function of diagonal 3D positive rook walks
(Bostan, Chyzak, van Hoeij, Pech, 2011):

$$\operatorname{Diag} f = \frac{1}{(2i\pi)^2} \oint \oint \frac{f(s, t/s, x/t)}{st} \, ds \, dt =$$

$$1 + 6 \cdot \int_0^x \frac{{}_2F_1\left(\begin{matrix} 1/3, 2/3 \\ 2 \end{matrix} \middle| \frac{27w(2-3w)}{(1-4w)^3}\right)}{(1-4w)(1-64w)} \, dw$$

$$\text{where} \quad f(s,t,u) = \frac{(1-s)(1-t)(1-u)}{1-2(s+t+u)+3(st+tu+us)-4stu}$$

Sums and Integrals (5/7)

q-Sums, e.g., from the theory of combinatorial partitions

Finite forms of the Rogers–Ramanujan identities and a generalisation: setting $(q;q)_n = (1-q)\cdots(1-q^n)$,

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \qquad \text{(Andrews, 1974)}$$

$$\sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i (q;q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \qquad \text{(Paule, 1985)}$$

Scalar products in algebraic combinatorics

For $p_1 = x_1 + x_2 + \cdots$ and $p_2 = x_1^2 + x_2^2 + \cdots$:

$$\left\langle \exp\left((p_1^2 - p_2)/2 - p_2^2/4\right) \,\middle|\, \exp\left(t\,(p_1^2 + p_2)/2\right) \right\rangle = \frac{e^{-\frac{1}{4}t\,(t+2)}}{\sqrt{1-t}}$$

Sums and Integrals (6/7)

Combinatorial identities

In the graph-counting sequence k^{k-1}:

$$\sum_{k=0}^{n} \binom{n}{k} i\,(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n \qquad \text{(Abel)}$$

In Stirling numbers of the second kind (partitions) and Eulerian numbers (ascents in permutations):

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle \qquad \text{(Frobenius)}$$

In Bernoulli numbers (Taylor expansion of $\tan(x)$):

$$\sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^{n} \binom{n}{k} B_{m+k} \qquad \text{(Gessel, 2003)}$$

Sums and Integrals (7/7)

Identities in more special functions (related, e.g., to number theory)

In Hurwitz's zeta function and the beta function:

$$\int_0^\infty x^{k-1} \zeta(n, \alpha + \beta x)\, dx = \beta^{-k} B(k, n-k)\, \zeta(n-k, \alpha)$$

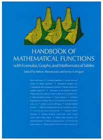
In the polylogarithm functions:

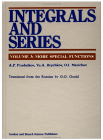
$$\int_0^\infty x^{\alpha-1} \operatorname{Li}_n(-xy)\, dx = \frac{\pi\, (-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$

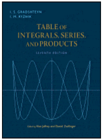
In the (upper) incomplete Gamma function:

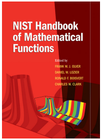
$$\int_0^\infty x^{s-1} \exp(xy)\, \Gamma(a, xy)\, dx = \frac{\pi y^{-s}}{\sin((a+s)\,\pi)} \frac{\Gamma(s)}{\Gamma(1-a)}$$

\+ a lot more in









or in web sites, like Victor Moll's web site on GR

Creative Telescoping for Sums and Integrals

$$U_n = \sum_{k=a}^{b} u_{n,k} = ?$$

Given a relation $a_r(n)u_{n+r,k} + \cdots + a_0(n)u_{n,k} = v_{n,k+1} - v_{n,k}$, summation leads by "telescoping" to

$$a_r(n)U_{n+r} + \cdots + a_0(n)U_n = v_{n,b+1} - v_{n,a} \overset{\text{often}}{=} 0.$$

$$U(t) = \int_a^b u(t,x)\,dx = ?$$

Given a relation $a_r(t)\frac{\partial^r u}{\partial t^r} + \cdots + a_0(t)u = \frac{\partial}{\partial x}v(t,x)$, integrating leads by "telescoping" to

$$a_r(t)\frac{\partial^r U}{\partial t^r} + \cdots + a_0(t)U = v(t,b) - v(t,a) \overset{\text{often}}{=} 0.$$

Adapts easily to $U(t) = \sum_{k=a}^{b} u_k(t),\ U_n = \int_a^b u_n(x)\,dx,$ etc.

Creative Telescoping for Sums and Integrals

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Telescoper Certificate

History of Algorithms for Creative Telescoping

Algorithmic Literature

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Applicable to

first-order vs higher-order equations; shift vs differential vs q-analogues vs mixed; ∂-finite vs non-∂-finite; w/ vs wo/ certificate.

Approaches

- bound denominators + bound degrees + linear algebra
- bound denominators + solve functional equations
- elimination by skew Gröbner bases/skew resultants
- reduction of singularity order + linear algebra

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Running Example

Problem

Integrate $f(n,p,x) = \dfrac{\exp(-px)T_n(x)}{\sqrt{1-x^2}}$ w.r.t. x and prove the identity

$$F(n,p) := \int_{-1}^{+1} f(n,p,x)\,dx = (-1)^n \pi I_n(p).$$

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$$F(n,p) := \int_{-1}^{+1} f(n,p,x)\,dx = (-1)^n \pi I_n(p).$$

Generating LFEs by algorithm for closure under product yields:

$$\frac{\partial f}{\partial p}(n,p,x) + xf(n,p,x) = 0,$$

$$nf(n+1,p,x) + (1-x^2)\frac{\partial f}{\partial x}(n,p,x) \\ + \big(p(1-x^2) - (n+1)x\big)f(n,p,x) = 0,$$

$$(1-x^2)\frac{\partial^2 f}{\partial_x^2}(n,p,x) - (2px^2+3x-2p)\frac{\partial f}{\partial_x}(n,p,x) \\ - (p^2x^2+3px-n^2-p^2+1)f(n,p,x) = 0.$$

Running Example

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Integrate $f(n,p,x) = \dfrac{\exp(-px)T_n(x)}{\sqrt{1-x^2}}$ w.r.t. x and prove the identity

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Compact notation using $f_n = f(n+1,p,x)$, $f_x = \frac{\partial f}{\partial x}(n,p,x)$, etc:

$$f_p + xf = 0,$$
$$nf_n + (1-x^2)f_x + \big(p(1-x^2) - (n+1)x\big)f = 0,$$
$$(1-x^2)f_{xx} - (2px^2+3x-2p)f_x - (p^2x^2+3px-n^2-p^2+1)f = 0.$$

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Observe: any $f_{n^u p^v x^w}$ is a $\mathbb{Q}(n,p,x)$-linear combination of f_x and f.

Running Example

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Goal: Find a telescoper such that there is a certificate

$$\sum_{u,v} c_{u,v}(n,p) f_{n^u p^v} = g_x \quad \text{for} \quad g = b(n,p,x)f_x + a(n,p,x)f.$$

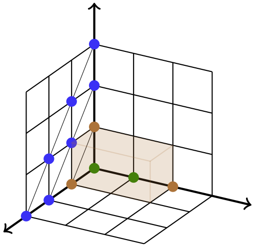
Chyzak's Algorithm (2000): an Example

$$\int_{-1}^{+1} f(n,p,x)\,dx = F(n,p) = ?$$

$$f_p = (\dots) f_x + (\dots) f,$$

$$f_n = (\dots) f_x + (\dots) f,$$

$$f_{xx} = (\dots) f_x + (\dots) f.$$



For $r = 1, 2, \dots$ until a nonzero equation can be found, solve:

$$\sum_{u+v \le r} c_{u,v}(n,p) f_{n^u p^v} = \frac{\partial}{\partial x}\big(b(n,p,x) f_x + a(n,p,x) f\big).$$

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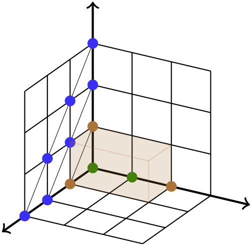

For $r = 1, 2, \dots$ until a nonzero equation can be found, solve:

$$\sum_{u+v\le r} (\dots)c_{u,v}(n,p)f_x + (\dots)c_{u,v}(n,p)f = \frac{\partial}{\partial x}\big(b(n,p,x)f_x + a(n,p,x)f\big).$$

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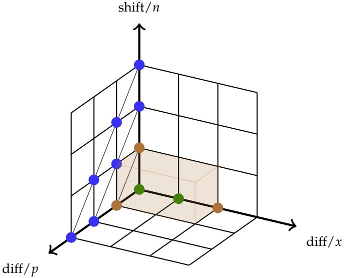

For $r = 1, 2, \dots$ until a nonzero equation can be found, solve:

$$\sum_{u+v\le r} (\dots) c_{u,v} f_x + (\dots) c_{u,v} f = ((\dots) b + b_x + a) f_x + ((\dots) b + a_x) f.$$

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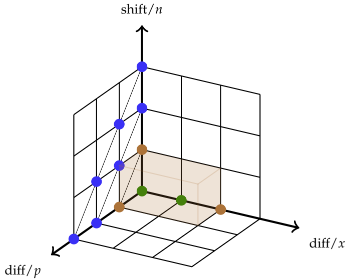

For $r = 1, 2, \dots$ until a nonzero equation can be found, solve:

$$\sum_{u+v\le r} (\dots)c_{u,v} = (\dots)b + b_x + a \quad \text{and} \quad \sum_{u+v\le r} (\dots)c_{u,v} = (\dots)b + a_x.$$

Chyzak's Algorithm (2000): an Example

$$\int_{-1}^{+1} f(n,p,x)\,dx = F(n,p) = ?$$

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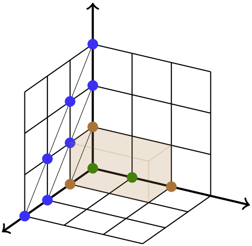

For $r = 1, 2, \dots$ until a nonzero equation can be found:

- eliminating a yields: $b_{xx} + (\dots)b_x + (\dots)b = \sum_{u+v\le r} (\dots)c_{u,v}$;
- a variant of Abramov's decision algorithm finds $b \in \mathbb{Q}(n,p,x)$ and the $(\dots)c_{u,v} \in \mathbb{Q}(n,p)$; substituting next gives a.

Chyzak's Algorithm (2000): an Example

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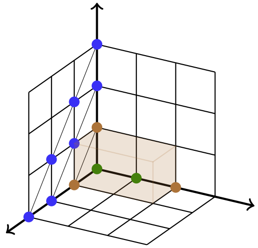

For the running example, the algorithm stops at $r = 2$ and outputs:

$$pf_p + pf_n - nf = g_x \quad \text{for} \quad ng = (1-x^2)f_x + \big(p(1-x^2) - x\big)f,$$
$$pf_{nn} - 2(n+1)f_n - pf = g_x \quad \text{for}$$
$$ng = 2x(1-x^2)f_x + 2\big((px+n)(1-x^2) - x^2\big)f.$$

Chyzak's Algorithm (2000): an Example

$$\int_{-1}^{+1} f(n,p,x)\,dx = F(n,p) = (-1)^n \pi I_n(p)$$

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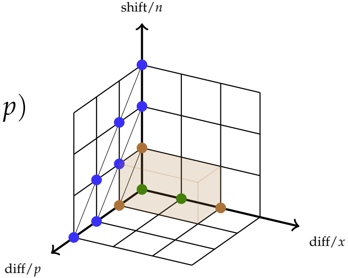

Upon integrating and using properties of $T_n(\pm 1)$:

$$pF_p + pF_n - nF = [g]_{x=-1}^{x=+1} = 0,$$
$$pF_{nn} - 2(n+1)F_n - pF = [g]_{x=-1}^{x=+1} = 0.$$

One recognizes the equations for the right-hand side.

```
[chyzak@slowfox (16:08:44) ~]$ maple -b Mgfun.mla -B
    |\^/|     Maple 2018 (X86 64 LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2018
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
> with(Mgfun);
[MG_Internals, creative_telescoping, dfinite_expr_to_diffeq,

    dfinite_expr_to_rec, dfinite_expr_to_sys, diag_of_sys, int_of_sys,

    pol_to_sys, rational_creative_telescoping, sum_of_sys, sys*sys, sys+sys]

> f := ChebyshevT(n,x)/sqrt(1-x^2)*exp(-p*x);
                            ChebyshevT(n, x) exp(-p x)
                       f := --------------------------
                                    2     1/2
                                (-x  + 1)

> ct := creative_telescoping(f, n::shift, p::diff, x::diff);
memory used=30.3MB, alloc=78.3MB, time=0.37
                           /d          \
ct := [[p _F(n + 1, p) + p |-- _F(n, p)| - n _F(n, p),
                           \dp         /

    x _f(n, p, x) - _f(n + 1, p, x)], [

    -p _F(n, p) + p _F(n + 2, p) + (-2 n - 2) _F(n + 1, p),

    -2 x _f(n + 1, p, x) + 2 _f(n, p, x)]]
```

Chyzak's Algorithm: Three Issues

1. The telescoper (wanted output) is a by-product of the certificate, which is obtained in dense, expanded form (likely to be unneeded in further calculations).
2. In dense, expanded form, the certificate is intrinsically large.
3. The rational-solving step is sensitive to r, allowing for little reuse of intermediate calculations.

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1. The telescoper (wanted output) is a by-product of the certificate, which is obtained in dense, expanded form (likely to be unneeded in further calculations).
2. In dense, expanded form, the certificate is intrinsically large.
3. The rational-solving step is sensitive to r, allowing for little reuse of intermediate calculations.

Example (walks in $\mathbb{N}^2$ using $\nwarrow, \leftarrow, \downarrow, \rightarrow, \nearrow$, counted by length):

$$\oint\oint \frac{-(1+x)(1+x^2-xy^2)}{(1+x^2)(1-y)(t-xy+ty+tx^2+tx^2y+txy^2)}\,dx\,dy$$

$$(16312320t^{20}+\cdots)f_{t^5}+(407808000t^{19}+\cdots)f_{t^4}+\ldots=\frac{\partial g}{\partial_x}+\frac{\partial h}{\partial_y}$$

LHS = 2 kB, g = 33 kB, h = 896 kB

1. Motivation
2. Creative telescoping and Chyzak's algorithm for D-finite functions (Chyzak, 2000)
3. Hermite reduction and the definite integration of rational functions (Bostan, Chen, Chyzak, Li, 2010)
4. Generalized Hermite reduction and the definite integration of D-finite functions (Bostan, Chyzak, Lairez, Salvy, 2018)
5. Conclusion

Rational Integration: the Classics

Hermite reduction

Given P/Q, Hermite reduction finds polynomials A and a such that

$$\int \frac{P(x)}{Q(x)}\,dx = \frac{A(x)}{Q^-(x)} + \int \frac{a(x)}{Q^*(x)}\,dx,$$

where $Q^*(x)$ is the squarefree part of $Q(x)$, $Q(x) = Q^-(x)Q^*(x)$, and $\deg a < \deg Q^*$.

Squarefree factorization

Given Q monic, one can in good complexity compute m and 2-by-2 coprime monic Q_i satisfying

$$Q = Q_1^1 Q_2^2 \dots Q_m^m, \quad Q^- = Q_2^1 \dots Q_m^{m-1}, \quad Q^* = Q_1 Q_2 \dots Q_m.$$

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Logarithmic part = obstruction to existence of rational primitive

For $R(w) = \mathrm{res}_x(b(x), a(x) - b'(x)w)$,

$$\int \frac{a(x)}{b(x)}\,dx = \sum_{R(c)=0} c \ln(\gcd(b(x), a(x) - b'(x)c))$$

(Trager, 1976).

Hermite Revisited (Bostan, Chen, Chyzak, Li, 2010)

$$F(t) := \oint \frac{P(t,x)}{Q(t,x)}\,dx = ? \qquad \text{ODE w.r.t. } t?$$

Hermite reduction in $K(x)$

Given P/Q, find polynomials A and a with $\deg a < \deg Q^*$ such that

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- Confinement $\deg_x a^{(i)} < d := \deg_x Q^* \leq \deg_x Q$:

$$\sum_{i=0}^{d} c_i(t)a^{(i)}(t,x) = 0 \implies \sum_{i=0}^{d} c_i F_{t^i} = 0.$$

- Incremental algorithm that does not compute $(P/Q)_{t^i}$.
- Degree bounds in $K(t)$ + eval./interpol. $\implies$ good complexity.

Operator Notation

Algebra of linear differential operators with rational coefficients

$$\mathbb{A} = K(t,x)\langle \partial_t, \partial_x \rangle, \qquad M_f = \mathbb{A}(f) = \{P(f) : P \in \mathbb{A}\}$$

$$P = \sum p_{i,j}(t,x)\partial_t^i \partial_x^j \in \mathbb{A} \implies P(f) = \sum p_{i,j}(t,x) f_{t^i x^j} \in M_f$$

$$\mathbb{S} = K(t,x)\langle \partial_x \rangle \subset \mathbb{A}$$

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Hypotheses of D-finiteness

- f is D-finite w.r.t. $\mathbb{A} \implies d := \dim_{K(t,x)}(M_f) < \infty$.
- Let $h \in M_f$ be cyclic, that is to say, $M_f = \bigoplus_{i=0}^{d-1} K(t,x) h_{x^i} = \mathbb{S}(h)$.
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Interpretation of creative telescoping

Given f, find a telescoper $T \in K(t)\langle\partial_t\rangle$ and a certificate $g \in M_f$ such that $T(f) = \partial_x(g)$. This really computes $\big(K(t)\langle\partial_t\rangle\big)(f) \cap \partial_x(M_f)$.

Lagrange's Identity

Dual of operators

$$P = \sum_{i=0}^{r} p_i(t,x)\partial_x^i \in \mathbb{S} \overset{*}{\longleftrightarrow} P^* = \sum_{i=0}^{r} (-\partial_x)^i p_i(t,x) \in \mathbb{S}$$

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Lagrange's identity

There is a map $\mathcal{L}_P$, bilinear w.r.t. $(h, \dots, h^{r-1})$ and $(u, \dots, u^{r-1})$, s.t.

$$\forall u \in K(t,x),\ \forall h \in M_f,\ uP(h) - P^*(u)h = \partial_x(\mathcal{L}_P(h,u)).$$

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Proof: $\mathcal{L}_P(h,u) = \sum_{i=0}^{r} \sum_{j=0}^{i-1} (-1)^j (p_i u)_{x^j} h_{x^{i-j-1}}$.

Consequences of Lagrange's Identity

Lagrange's identity:

$$\forall u \in K(t,x),\ \forall h \in M_f,\ uP(h) - P^*(u)h = \partial_x(\mathcal{L}_P(h,u)).$$

Let h be cyclic and $L \in \mathbb{S}$ be such that $L(h) = 0$. Then, for all $g \in M_f$:

Operator to rational function: $\mathbb{A}(f) = \mathbb{S}(h) \to K(t,x)h$

$g \in A_g^*(1)h + \partial_x(M_f)$.

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Equivalent rational factors: $K(t,x)h \to \big(K(t,x) \bmod L^*\big(K(t,x)\big)\big)h$

$\forall q \in K(t,x),\ g \in \big(A_g^*(1) - L^*(q)\big)h + \partial_x(M_f)$.

Consequences of Lagrange's Identity

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Running Example (continued)

Problem

Integrate $f(n,p,x) = \dfrac{\exp(-px)T_n(x)}{\sqrt{1-x^2}}$ w.r.t. x and prove the identity

$$F(n,p) := \int_{-1}^{+1} f(n,p,x)\,dx = (-1)^n \pi I_n(p).$$

In operator notation, f is cancelled by:

$$\partial_p + x1, \qquad n\partial_n + (1-x^2)\partial_x + \big(p(1-x^2) - (n+1)x\big)1,$$
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Goal: Find a telescoper such that there is a certificate

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$$\int_{-1}^{+1} f(n,p,x)\,dx = F(n,p) = ?$$

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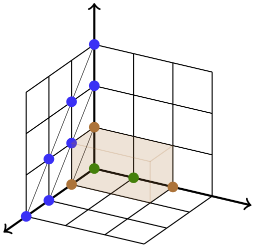

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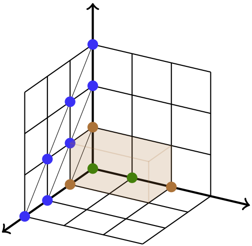

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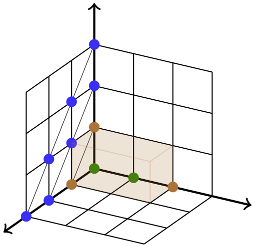

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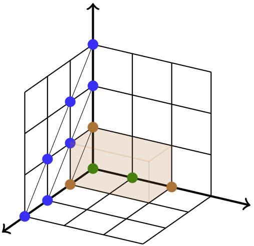

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For those P, $u - v_x \in K(p,n)[x]$ with degree ≤ 3, while

$$L^*(p^2x^0) = p^2x^2 - px - (n^2 + p^2),$$

$$L^*(p^2x^1) = p^2x^3 - 3px^2 - (n^2 + p^2 - 1)x + 2p.$$

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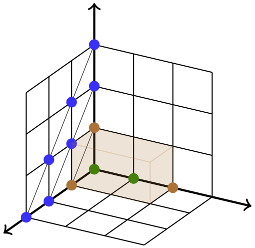

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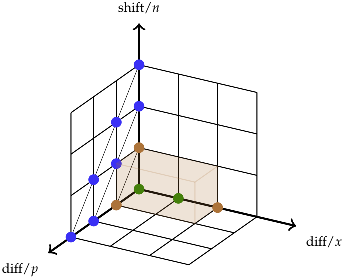

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Linear algebra over $K(p,n)$ finds a basis of telescopers

$$\left(\sum_P c_P P\right)(f) = \partial_x(\dots).$$

Reduction Modulo $L^*(K(x))$

Local decomposition of a rational function $R \in K(x)$

$R = R_{(\infty)} + \sum_{\alpha} R_{(\alpha)}$ for some $R_{(\alpha)} \in \frac{1}{x-\alpha}K(\alpha)[\frac{1}{x-\alpha}]$ and $R_{(\infty)} \in K[x]$.

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Local study of the action of L^*

$\exists$ polynomials I_α and I_∞, $\exists$ integers σ_α and σ_∞, such that $\forall s \in \mathbb{Z}$,

$$L^*\big((x-\alpha)^{-s}\big) = I_\alpha(-s)(x-\alpha)^{\sigma_\alpha - s} + \mathcal{O}((x-\alpha)^{\sigma_\alpha - (s-1)}) \text{ as } x \to \alpha,$$

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Weak reduction strategy

- reduce at finite αs (in any order) before at ∞,
- reduce term of higher valuation first (if possible),
- skip monomials for which $I_\alpha(-s-\sigma_\alpha) = 0$ or $I_\infty(-s-\sigma_\infty) = 0$.

Canonical Form Modulo $L^*(K(x))$

Problem: $L^*(K(x))$ does not reduce to 0

For $c_0 = I_\alpha(-s-\sigma_\alpha)$ and some c_1, write

$$L^*\big((x-\alpha)^{-s-\sigma_\alpha}\big) = c_0(x-\alpha)^{-s} + c_1(x-\alpha)^{-(s-1)} + \mathcal{O}((x-\alpha)^{-(s-2)}).$$

- If $c_0 \neq 0$, this reduces to
$$L^*\left((x-\alpha)^{-s-\sigma_\alpha} - (x-\alpha)^{-s-\sigma_\alpha}\right) = 0.$$
- If $c_0 = 0$, this reduces to some
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Solution

- finitely-many potential obstructions, described by the integer zeros of the I_α and I_∞,
- this can be computed, leading to a canonical-form computation.

```
[chyzak@slowfox (04:21:54) ~]$ maple -b Mgfun.mla -B
    |\^/|     Maple 2018 (X86 64 LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2018
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
> read "redct.mpl";
> f := ChebyshevT(n,x)/sqrt(1-x^2)*exp(-p*x);
                              ChebyshevT(n, x) exp(-p x)
                         f := --------------------------
                                        2     1/2
                                    (-x  + 1)

> redct(Int(f,x=-1..1),[n::shift,p::diff]);
memory used=3.5MB, alloc=8.3MB, time=0.09
                                             2
            [p D[n] + p D[p] - n, p D[n]  - 2 n D[n] - p - 2 D[n]]

> f := 2*BesselJ(m+n,2*t*x)*ChebyshevT(m-n,x)/sqrt(1-x^2);
                     2 BesselJ(m + n, 2 t x) ChebyshevT(m - n, x)
                f := --------------------------------------------
                                        2     1/2
                                    (-x  + 1)

> redct(Int(f,x),[t::diff, n::shift, m::shift]);
memory used=1189.8MB, alloc=144.8MB, time=9.98
                                              2
[t D[m] + t D[n] + t D[t] - m - n, t D[m]  - 2 m D[m] + t - 2 D[m],

          2
    t D[n]  - 2 n D[n] + t - 2 D[n]]
```

Timings: More than 140 integrals tested

Algorithm	(1)	(2)	(3)	(4)	(5)	(6)	(7)
new (mpl)	13s	> 1h	> 1h	1.5s	1.5s	165s	53s
Chyzak's (mma)	19s	253s	45s	232s	516s	>1h	>1h
Koutschan's (mma)	1.9s†	2.3s	5.3s	>1h	2.3s†	5.4s	2.2s†

$$\int \frac{2J_{m+n}(2tx)T_{m-n}(x)}{\sqrt{1-x^2}}\,dx \quad \text{[diff. } t\text{, shift } n \text{ and } m\text{]}, \tag{1}$$

$$\int_0^1 C_n^{(\lambda)}(x)C_m^{(\lambda)}(x)C_\ell^{(\lambda)}(x)(1-x^2)^{\lambda-\frac{1}{2}}\,dx \quad \text{[shift } n, m, \ell\text{]}, \tag{2}$$

$$\int_0^\infty xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx \quad \text{[diff. } a\text{]}, \tag{3}$$

$$\int \frac{n^2+x+1}{n^2+1}\left(\frac{(x+1)^2}{(x-4)(x-3)^2(x^2-5)^3}\right)^n \sqrt{x^2-5}\,e^{\frac{x^3+1}{x(x-3)(x-4)^2}}\,dx \quad \text{[shift } n\text{]}, \tag{4}$$

$$\int C_m^{(\mu)}(x)C_n^{(\nu)}(x)(1-x^2)^{\nu-1/2}\,dx \quad \text{[shift } n, m, \mu, \nu\text{]}, \tag{5}$$

$$\int x^\ell C_m^{(\mu)}(x)C_n^{(\nu)}(x)(1-x^2)^{\nu-1/2}\,dx \quad \text{[shift } \ell, m, n, \mu, \nu\text{]}, \tag{6}$$

$$\int (x+a)^{\gamma+\lambda-1}(a-x)^{\beta-1}C_m^{(\gamma)}(x/a)C_n^{(\lambda)}(x/a)\,dx \quad \text{[diff. } a\text{, shift } n, m, \beta, \gamma, \lambda\text{]}. \tag{7}$$

†: Heuristic got these faster answers by looking for telescopers of non-minimal orders, yet smaller sizes.

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Need to investigate failures: non-mathematical bugs? “not ours”?