TOWARDS CORRECTLY ROUNDED MIXED-RADIX IEEE754 ARITHMETIC

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Motivations

```
int main() {
  _Decimal128 a = 0.1D;
  float b = 10.25;
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- the result is $\mathtt{d} = \mathtt{0x1p} - 52 \approx 2.2204 \cdot 10^{-16}$
- as a reminder, the smallest subnormal number is $\mathtt{0x1p} - 1074 \approx 4.9407 \cdot 10^{-324}$

IEEE 754-2008 - FP formats

Binary format

$(-1)^s \cdot 2^E \cdot m$

s	$E + \text{bias}$	m
1 bit	W_E bits	$p - 1$ bits

Example, `binary` formats:

- significand: $2^{k_2 - 1} \leq m \leq 2^{k_2} - 1$
- exponent: $E_{\text{min}} \leq E \leq E_{\text{max}}$
  (with subnormals)

with $k_2 = p$.

Decimal format

$(-1)^s \cdot 10^F \cdot n$

s	$F + \text{bias}$	n
1 bit	$w + 5$ bits	$10 \times J$ bits

Example, `decimal`$\{k\}$ format:

- significand: $1 \leq n \leq 10^{k_{10}} - 1$
- exponent: $F_{\text{min}} \leq F \leq F_{\text{max}}$
- binary BID encoding

with $k_{10} = 9 \times k/32 - 2$.

IEEE 754-2008 - Rounding Modes

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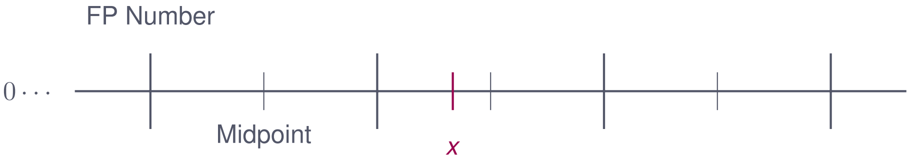

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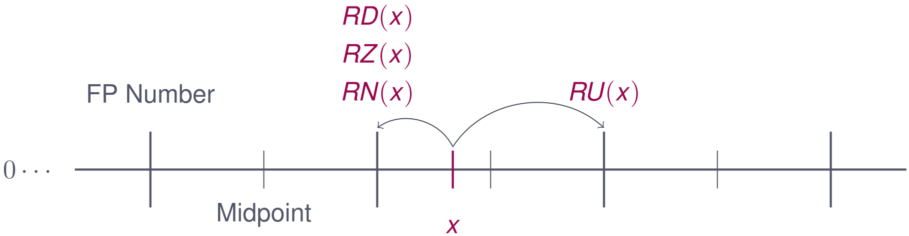

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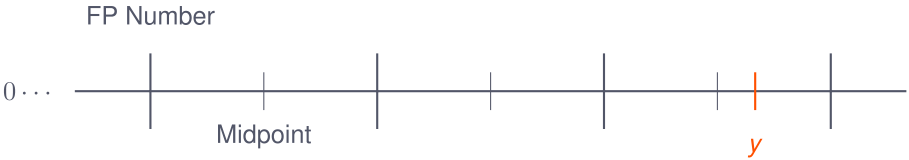

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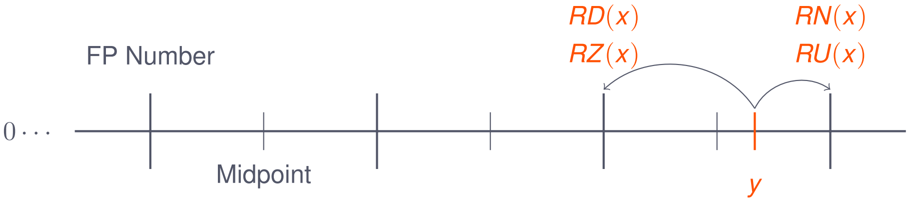

IEEE 754-2008 - Arithmetic Operations

Definitions and properties

- basic arithmetic operations ($+$, $\times$, $\div$, *FMA*...)
- exceptions and flags
- heterogenous operations
  - same base, different format/precision
  - e.g. $\texttt{binary32} = \texttt{binary32} \times \texttt{binary64}$

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Goal: mixed-radix operations

Enrich the IEEE 754-2008 standard with heterogenous operations in base 2 and 10.

Designing Mixed-Radix Operations

Considered formats

- Binary: `binary32`, `binary64`, `binary128`
- Decimal: `binary64`, `binary128`

Considering all basic operations: ≈ 1120 **operations**

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Goal: Automatically generate mixed-radix basic operations

Compromise between efficiency and implementation effort.

Designing Mixed-Radix Operations

Definition: Fused Multiply and Add

$$\texttt{FMA}(a, b, c) = \circ(a \times b + c) \qquad \texttt{where } \circ \in \{\texttt{RN}, \texttt{RZ}, \texttt{RU}, \texttt{RD}\}$$

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Goal: mixed-radix FMA

- an emerging need for mixed-radix arithmetic
- implementation of all basic arithmetic operations with one slightly more precise FMA

Table Maker's Dilemma

Example: consider the exact transcendental number $y = e^x$ and the computed result $\widehat{y} = \exp(x)$.

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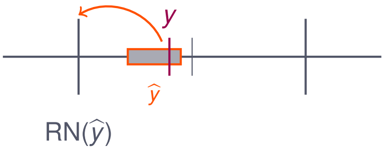

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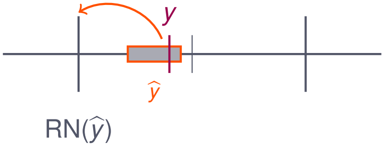

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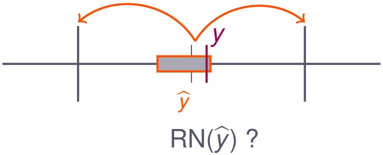
Correct Rounding in the hard case



- not enough accuracy, but how much?

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Classical Binary FMA

Algorithm 1 Binary FMA $d = \circ(a \times b + c)$

1: **if** $\frac{a \times b}{c} \notin \left[\frac{1}{2}, 2\right]$ **then**
2: d = *farpath*_addition($a \times b$, c)
3: **else**
4: d = *nearpath*_subtraction($a \times b$, c)
5: **end if**

far-path addition

- when $\frac{a \times b}{c} \notin \left[\frac{1}{2}, 2\right]$
- simple logic with sticky guard bit

near-path subtraction

- when $\frac{a \times b}{c} \in \left[\frac{1}{2}, 2\right]$
- **Sterbenz's lemma**: $(a \times b) - c$ is exactly representable

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Obervation

Mixed-radix addition almost always inexact.

Overcoming the TDM - Mixed-Radix unified format

Obervations

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$$k_2' = \max(k_2 + 1, \lceil \log_2(10^{k_{10}} - 1) \rceil + 1), \Lambda_5^{\min} = \left\lceil \log_5 \left(\frac{1}{2^{k_2'} - 1} \right) \right\rceil \text{ and } \Lambda_5^{\max} = \left\lfloor \log_5 \left(\frac{1}{2^{k_2'-1}} \right) \right\rfloor.$$

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At a certain precision, binary to decimal conversion becomes exact.

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Bound on the worst case of cancellation

- occurs when $(a \times b) - c$ is relatively small
- if $a \times b = 2^L \cdot 5^M \cdot s$ and $c = 2^N \cdot 5^P \cdot t$

$$\left| \frac{s}{t} - 2^{N-L} \cdot 5^{P-M} \right| \geq \eta$$

- computed using one sided approximations

	η
binary64, decimal64	$2^{-177.61}$
binary128, decimal128	$2^{-360.25}$

Performances issues of this exact addition

Size of the long accumulator

- acutal computation $\alpha = (a \times b) + c - f$
- a, b and c inputs of the FMA, $(a \times b)$ the exact multiplication bounded into the internal mixed-radix format
- f the closest midpoint bounded into the internal mixed-radix format

	AC^{max} (bits)	AC^{max} (words)	free bits
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Obervation

In a lot of cases, a quick and not so accurate addition can be enough to perform correct rounding in the output format.

FMA Mixed Radix Algorithm

Algorithm 2 Mixed-Radix FMA $d = \circ(a \times b + c)$

```
Multiplication ψ ← a × b
if it is an "addition" or ψ/c ∉ [1/2; 2] then
    φ ← "far-path" binary addition
else
    φ ← "near-path" binary subtraction
end if
ρ ← Conversion of φ to the output format
if ρ can round correctly then
    return  d ← ρ correctly rounded to output format
else
    Compute integer rounding boundary significand f
    α ← Exact decimal addition
    Correct ρ using f and the sign of α
    return  d ← ρ correctly rounded to output format
end if
```

Test Environment and Reference implementations

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GNU Multiple Precision Library (GMP)

- mixed-radix FMA designed in a limited timeframe
- using GMP rational numbers
- Goal: reasonabely fast but easy to design

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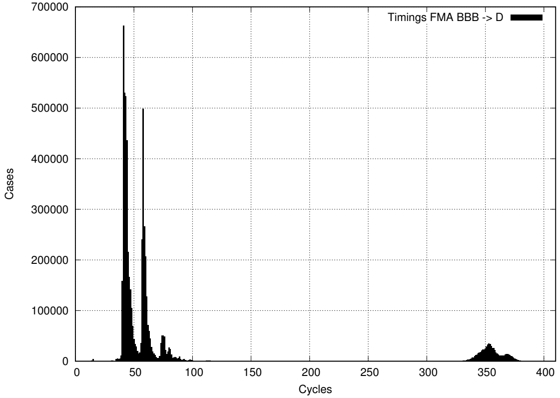
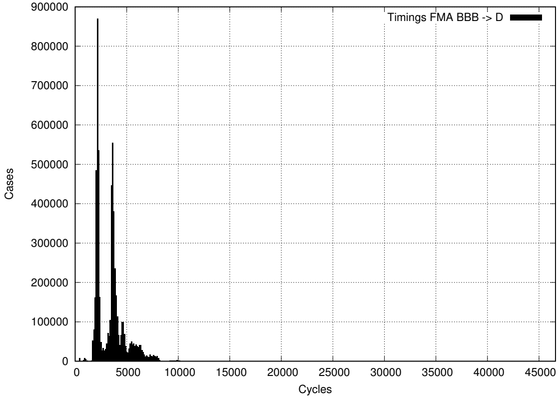
- exact representation of numerical expressions
- evaluated at any precision without spurious rounding

Performance Testing



Our implementation



GMP reference implementation

Conclusion and Perspectives

Correctly Rounded Mixed-Radix FMA

- two formats: `binary64` and `decimal64`
- pen and paper proof of the algorithm
- overcoming the TDM and worst case of cancellation in the mixed-radix case
- implementation faster than expected and extensively tested

Going further

- finalize code generator implementation
- optimize FMA for heterogenous precision

Thank you! Questions?