Continued Logarithm Algorithm: A probabilistic study

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Work with

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The origins

Introduced by Gosper as a mutation of continued fractions:

- gives rise to a gcd algorithm akin to Euclid's.
- quotients are powers of two:
  - small information parcel.
  - employs only shifts and substractions.
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More recently:

- Shallit studied its worst-case performance in 2016.
- We consider its average performance!

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Example. Let us find $\gcd(13, 31)$.

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- Ended with $(0, 8)$, what is the $\gcd$? $\Rightarrow$ parasitic powers of 2.

Consider

$$\Omega_N = \{(p,q) \in \mathbb{N} \times \mathbb{N} : p \leq q \leq N\}.$$

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Main result [RVV18].

Mean number of **steps** $E_N[K]$ and **shifts** $E_N[S]$ are $\Theta(\log N)$.

More precisely

$$E_N[K] \sim k \log N, \qquad E_N[S] \sim \frac{\log 3 - \log 2}{2\log 2 - \log 3} E_N[K]$$

for an *explicit constant* $k \doteq 1.49283\ldots$ given by

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$$k = \frac{2}{H} , \quad H = \tfrac{1}{\log(4/3)} \left(\tfrac{\pi^2}{6} + 2 \sum_j \tfrac{(-1)^j}{2^j j^2} - (\log 2) \tfrac{\log 27}{\log 16} \right)$$

Process depends **only** on p/q rather than (p, q).

- Map $p/q \mapsto p'/q'$ can be extended to $\mathcal{I} = (0, 1)$

$$T\colon \mathcal{I} \to \mathcal{I}\,, \;\; T(x) = \frac{1}{2^a x} - 1\,,$$

where $a = \lfloor \log_2(1/x) \rfloor$.

- Iteration gives a special continued fraction

$$\frac{p}{q} = \frac{1}{2^a \left(1 + \frac{p'}{q'}\right)}.$$

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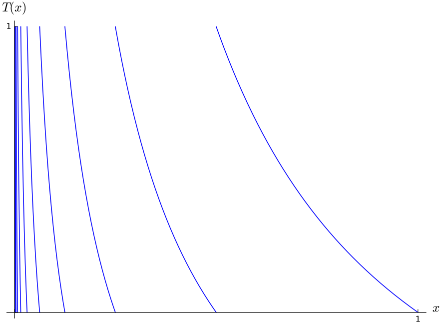
The continued fraction expansion ends (is finite) when we get 0.

The CL dynamical system [Chan05]



The map $T\colon \mathcal{I} \to \mathcal{I}$

Branches

For $x \in \mathcal{I}_a := [2^{-a-1}, 2^{-a}]$

$$x \mapsto T_a(x) := \frac{2^{-a}}{x} - 1\,.$$

where $a(x) := \lfloor \log_2(1/x) \rfloor\,.$

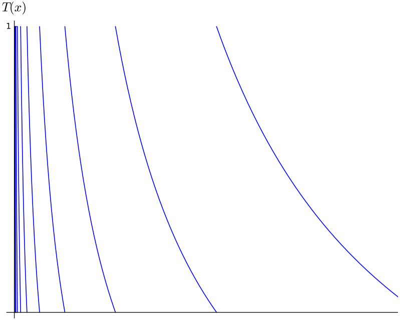
Inverse branches

$$h_a(x) := \frac{2^{-a}}{1+x}\,, \quad \mathcal{H} := \{h_a : a \in \mathbb{N}\}\,,$$

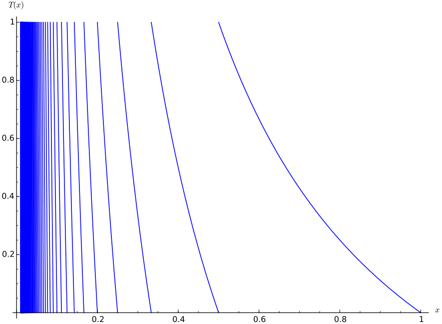
and at depth k

$$\mathcal{H}^k := \{h_{a_1} \circ \cdots \circ h_{a_k} : a_1, \ldots, a_k \in \mathbb{N}\}\,.$$

Dynamical system $(\mathcal{I}, T)$



The map for the CL algorithm



The map for Euclid's algorithm.

Density transformer

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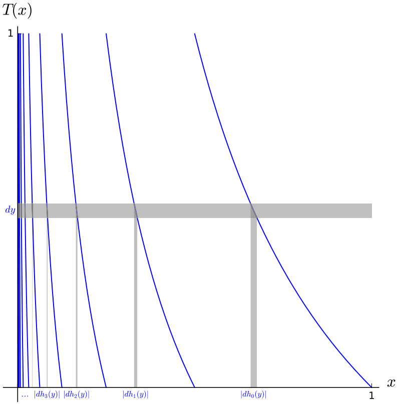

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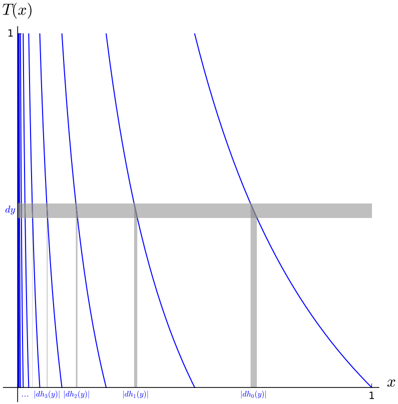

Answer: The density is

$$\mathbf{H}[g](x) = \sum_{h \in \mathcal{H}} \left| h'(x) \right| g\left(h(x)\right)$$
$$= \frac{1}{(1+x)^2} \sum_{a \geq 0} 2^{-a} g\left(\frac{2^{-a}}{1+x}\right).$$

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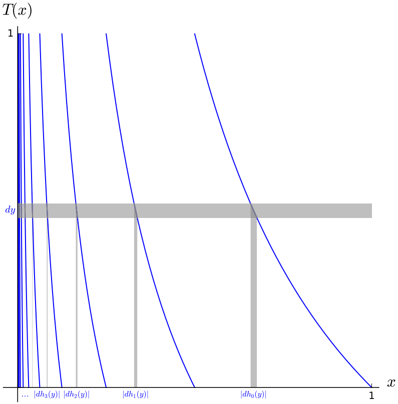

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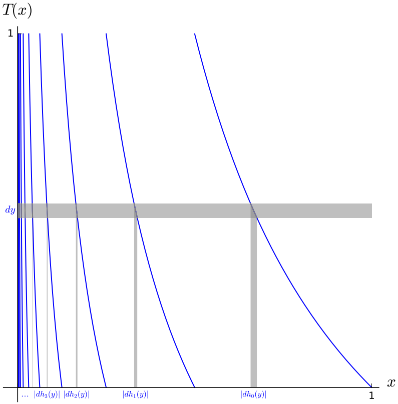

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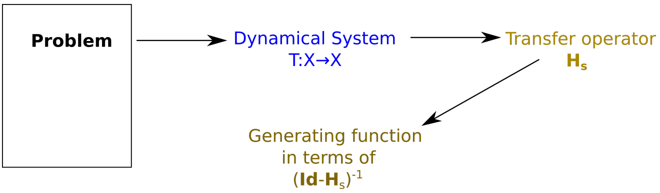
$\implies$ Transfer operator $\mathbf{H}_s$ extends $\mathbf{H}$, introducing a variable s

$$\mathbf{H}_s[g](x) = \sum_{h \in \mathcal{H}} \left| h'(x) \right|^s g\left(h(x)\right) .$$

Principles of dynamical analysis [Vallée,Flajolet,Baladi,...]:

Generating functions.

- $\mathbf{H}_s$ describes all executions of depth 1.
- $\mathbf{H}_s^2 = \mathbf{H}_s \circ \mathbf{H}_s$ describes all executions of depth 2.
- $\vdots$
- and $(\mathbf{I} - \mathbf{H}_s)^{-1} = \mathbf{I} + \mathbf{H}_s + \mathbf{H}_s^2 + \ldots$ describes *all* executions.



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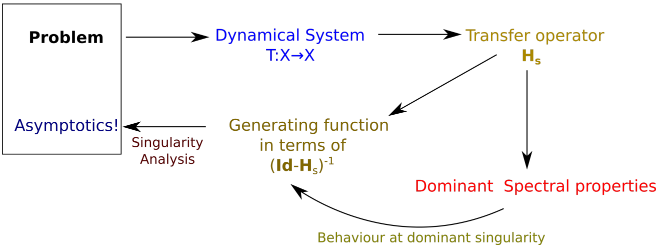

Reduced denominators and inverse branches

Euclidean algorithm:

- Homographies

$$h_m(x) = \frac{1}{m+x},$$

with $\det h_m = -1$.

- For $h = h_{m_1} \circ \cdots \circ h_{m_k}$

$$h(0) = \frac{p}{q} \Rightarrow |h'(0)| = \frac{1}{q^2},$$

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Problem: Denominator retrieved is engorged by powers of two.

Recording the dyadic behaviour

Solution: Dyadic numbers $\mathbb{Q}_2$!

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Idea works!

The extended dynamical system

- Introduce $\underline{\mathcal{I}} := \mathcal{I} \times \mathbb{Q}_2$ and $\underline{T}\colon \underline{\mathcal{I}} \to \underline{\mathcal{I}}$ as follows

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  Haar (translation invariant) measure ν on $\mathbb{Q}_2$ has one!

Functional space $\mathcal{F}$ for the extended operator $\underline{\mathbf{H}}_s$

Real component directs the dynamical system:

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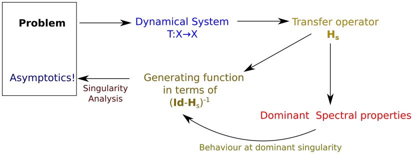
We can finish the dynamical analysis!



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Back to $(13, 31)$

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